UUCMS. No.

B.M.S COLLEGE FOR WOMEN, AUTONOMOUS

BENGALURU – 560004

SEMESTER END EXAMINATION – SEPT/OCT-2023

M.Sc in Mathematics – 4th Semester

MEASURE AND INTEGRATION

Course Code: MM401T **QP Code: 14001**

Duration: 3 Hours **Max marks: 70**

Instructions: ***1) All questions carry equal marks.***
2) Answer any five full questions.

1. a) Let E be a given set. Then prove that the following statements are equivalent.
   (i) E is measurable.
   (ii) Given $\epsilon > 0$, there is an open set $O \supset E$ such that $m^*(O - E) < \epsilon$.
   (iii) There is a G_δ set $G \supset E$ such that $m^*(G - E) = 0$.
   b) Show that Cantor ternary set has measure zero.

(7+7)

2. a) Prove that (a, ∞) is measurable.
   b) Prove that union of two measurable set is measurable.
   c) If E is a measurable set and $y \in \mathbb{R}$, then prove that $E + y$ is measurable.

(6+4+4)

3. a) If f, g are measurable functions on a measurable set E and c is any real number, then prove that $cf, f + g$ and fg are measurable.
   b) Let f be a function defined on a measurable set E. Then prove that f is measurable if and only if for any open set G in $\mathbb{R}, f^{-1}(G)$ is measurable.

(8+6)

4. a) Let $f \geq 0$ be measurable. Then show that there exists a sequence $\{\phi_n\}$ of simple functions such that $\phi_n \to f$.
   b) Let E be a measurable set with $m(E) < \infty$ and $\{f_n\}$ be a sequence of measurable functions defined on E. Let f be a real-valued measurable function defined on E such that $f_n(x) \to f(x)$ for each $x \in E$. Then prove that for given $\epsilon > 0$ and $\delta > 0$, there is a measurable set $A \subset E$ with $m(A) < \delta$ and an integer N such that $| f_n(x) - f(x)| < \epsilon, \forall x \in E - A$ and $\forall n \geq N$.

(7+7)

5. a) Let f be a bounded function defined on a measurable set E. Then show that f is measurable if and only if f is Lebesgue integrable.
   b) State and prove bounded convergence theorem.

(8+6)

6. a) State and prove Lebesgue dominated convergence theorem.

b) If f and g are non-negative measurable functions defined on E, then prove that

$$(i) \int_E cf = c\int_E f \text{ for any constant } c \geq 0 \text{ and } (ii) \int_E (f+g) = \int_E f + \int_g g$$

(iii) If $f \leq g$ a.e, then $\int_E f = \int_E g$.

(6+8)

7. a) Prove that every bounded monotone function is a function of bounded variation and every absolutely continuous function is a function of bounded variation.

b) Let f be an increasing real-valued function defined on $[a, b]$. Then show that f is differentiable almost everywhere, the derivative f' is measurable and

$$\int_a^b f'(x)dx \leq f(b) - f(a).$$

(6+8)

8. a) Let f be an integrable function defined on $[a, b]$. If $\int_a^x f(t)dt = 0$ for all $x \in [a, b]$. Then prove that $f = 0$ almost everywhere on on $[a, b]$.

b) Let f be an absolutely continuous function on $[a, b]$. Then show that f is an indefinite integral of some integrable function.

(7+7)

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